On a multi item integrated production inventory systems with variable parameters

ZAID BALKHI

Istanbul University

Faculty of Forestry

Bahceköy , Sariyer

Istanbul Turkey

E-Mail: ztbalkhi@gmail.com Web : www.orman.istanbul.edu.tr

Abstract .*Most of the classical production inventory systems treat the economic order quantities (EOQ) of raw materials, which are used in producing the same final products, separately of the economic production quantities (EPQ) of these products. This may result in sub-optimization of both the (EOQ)'s of raw materials and the (EPQ)'s of final products. But, when raw materials are used in production, the ordering quantities for raw materials are dependent on the economic batch size and the schedule of the final products. By integrating the procurement and production subsystems, the degree of sub-optimality is reduced. In this paper, a unified inventory model of integrated production inventory systems, where each of production, demand and deterioration rates of final products and deterioration rates of raw materials , as well as all cost and profit parameters are general functions of time . Shortages are allowed only for final products but are partially backordered. All cost and profit components are affected by both inflation and time value of money. The objective is to find an optimal production schedule for each product in any inventory cycle so that the overall total relevant inventory net profit for this integrated system is maximized. We develop an exact formula for the total net profit per unit of time. Then, we use rigorous mathematical methods to find the optimal stopping and restarting times for each final product of this integrated inventory systems.*

Keywords. Multi-item integrated systems, Deterioration, Inventory control, Optimality.

1. Introduction

In so many industrial systems like petrochemicals, wood extracted from forests, the same kinds of raw materials are used in producing tens of final products . This means that the ordering quantities for raw materials are dependent on the economic batch sizes of the resulting final products ,hence on the schedules of the final products. In fact, most of the classical production inventory systems treat the economic order quantities (EOQ)'s of raw materials, which are used in producing certain final products, separately of the economic production quantities (EPQ) of these products. This may result in sub-optimization of both the (EOQ)'s of raw materials and the (EPQ)'s of final products. By integrating the procurement and production subsystems, the degree of sub-optimality as well as the total relevant net profit of the corresponding system is maximized. Applying such research results are expected to save huge amounts of money that can be used for development, as it is the case in most first class countries.. For example and according to Nahmias Book (Production and Operations Analysis (1997)), the investment in inventories in the United States held in the manufacturing, wholesale and retail sectors during the first quarter of 1995 was estimated to be $1.25 trillion**.** Therefore, there is a great need to perform special research on inventory control management, in particular for such giant integrated inventory systems, in order to improve their efficiency and performance in such a way that their total relevant net profit is maximized.

Nevertheless, many classical inventory researchers concern with single item and with the so called economic order quantity (EOQ) models. problem,

Among these are Cardenas and Barron [18] ,Resh et al [33], Chang [19], Yan and Cheng [37] ,Dye et al. [22], Chen and Chen [20] , Ben-Daya et al[15], Manna et al [27] and, Maiti and Maiti [28]. A more dynamic inventory model was presented by Balkhi and Tadj [7], where they derived an EOQ model with deteriorating items and time varying demand, deterioration, and costs. Balkhi [6] conducted another study in which he treated the variability of parameters of an inventory model for deteriorating items under trade credit policy with partial backordering and an infinite time horizon. More classical inventory models concern with single item and with the (EPQ) models has also been introduced by several authors. Darwish [21] generalized the classical (EPQ) model by studying the relation between the setup cost and the length of the production cycle. An inventory model in which products deteriorate at a constant rate and in which demand, and production rates are allowed to vary with time has been introduced by Balkhi and Benkhrouf [14]. Subsequently , Balkhi [3], [8],[9],[11],[13] and Balkhi et al [10] , have introduced several (EPQ)inventory models in each of which , the demand , production , and deterioration rates are arbitrary functions of times ,and in some of which , shortages are allowed but are completely backlogged. In each of the last mentioned six papers, closed forms of the total inventory cost were established, and the conditions that guarantee the optimality of the solution for the considered inventory system were also introduced.

Concerning multi items (EPQ) models ,Ben-Daya and Raouf [16] have developed an approach for more realistic and general single period for multi-item with budgetary and floor or shelf space constraints, where the demand of items follows a uniform probability distribution subject to the restrictions on available space and budget. Bhattacharya [17] has studied two-item inventory model for deteriorating items. Lenard and Roy [26] have used different approaches for the determination of optimal inventory policies based on deteriorating items with constraint space and investment. Rosenblatt [32] has discussed multi-item inventory system with budgetary constraint using the comparison between the Lagrangean and the fixed cycle approach, whereas, Rosenblatt and Rothblum [33] have studied a single resource capacity where this capacity was treated as a decision variable. Balkhi and Foul [4] and [5] have treated the problem of multi-item inventory control but without resource constraints. Recently, Balkhi [2] ,has introduced a general multi item production inventory system under budgetary constraint in which the parameters of the model including the cost parameters are all arbitrary functions of time. For more details about multi-item inventory system, the readers are advised to consult the survey of Yasemin and Erenguc [38] and the references therein.

However, most of the classical production inventory papers even those concerned with multi items treated the economic order quantity (EOQ) of raw materials, which are used in producing certain final products, separately of the economic production quantity of these products. This may result in sub-optimization of both the (EOQ) of raw materials and the (EPQ) of the final products. Therefore, a model which unifies the optimization for raw materials and final products is preferable. To cite a few examples of research papers on such models, Goyal [23] may be the first contributor to this area .He considered an integrated inventory system for a single product with non deteriorating items. Also, integrated production inventory systems are subject to deterioration if they are in the situation of work-in-process (WIP). Pahl et al. [29] survey the recent trends in modeling deterioration in the various fields of production planning and give an extensive overview of the subject. Park [30] studied a production inventory system for a single product with deteriorating raw materials. Raafat [31] generalized Park's model to include deterioration of final product. Raafat [32] extended his earlier model by taking raw material and final product both subject to a specific monotonic increasing deterioration rate. Balkhi [12] generalized the model of Raafat[32] by assuming time dependence of the production rate, demand rate for final product, and of deterioration rates for both final product and raw materials. Law and Wee [25] considered an integrated production-inventory model with both ameliorating and deteriorating effects in which the manufacturers treat livestock as the raw materials. They took in to account multiple deliveries, partial backordering and time discounting. The amelioration and deterioration rates are assumed to follow the Weibull distribution .For a complete treatment of inventory theory and models the reader is referred to Zipkin [39].

The main goal of this paper is to generalize the paper of Balkhi [1] to the case where all cost parameters are general functions of time, and to generalize the integrated production inventory systems of several products ,which share the same raw materials, in various ways. First, each of the production ,demand and deterioration rates of final products and the deterioration rates of raw materials as well as all cost and profit parameters are assumed to be general functions of time instead of being a linear or constant functions. Second, shortages are allowed only for each of the final products, so that part of these shortages is backlogged and the rest are lost. The part of the shortage that is backlogged is proportional to the waiting time. Third ,the on-hand inventories are affected by two kinds of deterioration. The first kind is the one that has been widely used and concerns items deterioration with time while are effectively in stock. The second one is concerned with (WIP). Forth, we consider the effect of inflation and the time value of money on each cost and profit component. Fifth, this paper deals with a very general (EPQ) integrated inventory model for multi items that use the same raw materials in producing certain final products and for which many of the related models available in the literature are special cases of the introduced model. Our analysis deepens, broadens, and enriches the available theoretical studies; in particular the mathematical results related to (EPQ) integrated production inventory models of multi items.

The rest of the paper is organized as follows. Following this introduction, we introduce the model assumptions and notations. The problem is formulated in section 3. We introduce a solution procedure for the problem in section 4. And in sections 5 and 6, we provide sufficient conditions that guarantee the optimality and the uniqueness of any existing solution. We finally conclude the paper in section 7.

2. Model Assumptions and Notation

The integrated production inventory model proposed in this paper is based on the following assumptions and notation.

1. A number ***m*** of raw materials is required to manufacture ***n*** final product.

2. Deterioration occurs when items are effectively in stock for both raw materials and final products and there is no repair or replacement of deteriorated items during any given cycle. However, the (WIP) deterioration is possible for the final products.

3. The demand, production, and deterioration rates of the final product ***i*** are known and general functions of time, denoted by $D_i(t)$; $P_i(t)$, and $\theta_i(t)$, respectively , $i=1,2,\ldots,n$.

4. The raw materials can be ordered or produced from outside suppliers at infinite rate of replenishment so that the arrival of all needed raw materials coincides with the start of a new production run of the final products.

5. Raw material ***j*** is subjected to deterioration and its deterioration rate is a known and general function of time denoted by $\delta_j(t)$; $j = 1,2,\ldots,m$.

6. r_{ij}: is the amount of raw material ***j*** to make one unit of the final product ***i***.

7. Shortages are allowed for the final products but only a fraction $\beta_i(\tau_i) = e^{-\tau_i}$ ($0 \leq \beta_i(\tau_i) = e^{-\tau_i} \leq 1$) , of the final product i is backordered and the remaining fraction (1-$\beta_i(\tau_i)$) is lost, where $\tau_i = T_{i3} - t$ (see Fig. 1 below):

is the waiting time up to the new production of the final product i when shortages for this product start to be backlogged by time T_{i3}. Note that, $\beta_i(\tau_i)$ is a decreasing function of τ_i, which reflects the fact that less waiting time implies more backordered item of final product ***i***.

8. Shortages are not allowed for any of the raw materials and lead time is negligible.

9. The stopping and restarting times are denoted as follows (we refer to Fig. 1):

T_{i1} : time at which the production for the final product i stops.

T_{i2} : time at which the inventory level for the final product i reaches zero.

T_{i3} : time at which the production for the final product i restarts.

T_{i4} : the end of the cycle for the final product i .

10 .$I_i(t)$ is the inventory level of the final product i and $q_j(t)$ is the inventory level of raw material j , $i = 1, 2,\ldots,n$, and $j = 1, 2,\ldots,m$.

11. The cost structure of this integrated production inventory system is as follows:

(a) ***for the final product i***

i. $h_i(t)$: inventory holding cost per unit per unit time .

ii. $s_i(t)$: setup cost per cycle.

iii. $c_i(t)$ item (production) cost per unit.

iv. $b_i(t)$:shortage cost per unit per unit of time for backlogged items.

v. $l_i(t)$ shortage cost per unit per unit of time for lost items.

Vi. $n_i(t)$ is the unit profit of non deteriorated items from the final product ***i***

(b) ***for raw material j; j = 1,…..,m***

i. o_j (t): inventory holding cost per unit of raw material j per unit of time.

ii. k_j (t) order cost per order.

iii. u_j (t) unit cost.

12. All costs and profits components are affected by inflation rate and time value of money. We shall denote by d_1 the inflation rate and by d_2 the discount rate representing the time value of money so that $r = d_2 – d_1$ is the discount rate net of inflation

The objective is to maximize the total relevant net profit per unit time for the final products.

The proposed system operates as follows. The cycle starts at time $t = 0$ and the inventory of the final product i accumulates at a rate $P_i(t)$-$D_i(t)$- $\theta_i(t)I_i(t)$ up to time $t= T_{i1}$ where the production of this product stops. Meanwhile, the inventory level of raw material j decreases at a rate $r_{ij}P_i(t)$- $\delta_j(t)\, q_j(t)$. After that, the inventory level of the final product i starts to decrease due to demand and deterioration at a rate $D_i(t)$ - $\theta_i(t)I_i(t)$ up to time $t = T_{i2}$, where shortages start to accumulate at a rate $\beta_i(\tau_i)\, D_i(t)$ up to time $t=T_{i3}$ .Production for the final product i restarts again at time $t = T_{i3}$ and ends at time $t = T_{i4}$ with a rate $P_i(t)$-$D_i(t)$ to recover both the previous shortages in period $[T_{i2},T_{i3}]$ and to satisfy demand in period $[T_{i3},T_{i4}]$, whereas the inventory level of raw material j decreases at a rate $r_{ij}P_i(t)$-$\delta_j(t)\, q_j(t)$ during $[T_{i3},T_{i4}]$. The process is repeated. In this respect and in order to recover the backordered items within period $[T_{i2},T_{i3}]$ and to satisfy the demand in period $[T_{i3},T_{i4}]$. In this respect we require that.

$$P_i(t) > [1+\beta_i(T_{i3}-t)]D_i(t) = [1+\beta_i(\tau)]D_i(t)\,, i = 1,2.....,n$$

Though the deterioration of raw materials has its own effects on the deterioration of the final products, but it is not necessary to include such effects in the production rate of the final products since we shall treat the last as arbitrary functions of time. On the other hand, since the decisions of stopping and restarting of the production of any final product occur in a single cycle and are only related to the previous or next cycle through its initial inventory level, it is only necessary to consider the problem in a single cycle.In this respect, and to maintain the repeat of the process, we shall assume the following (which we shall refer to as assumption 1, or AS1.)

The raw materials in the first stages of production must cover the need of these raw materials

for the final products exactly. Similar thing must hold for the second stage of production (AS1.)

The behavior of the underlying inventory system is shown in Fig 1. below.

3. Problem Formulation

The changes of the inventory level $I_i(t)$ for the final product *i* is governed by the following differential equations

$$\frac{dI_i(t)}{dt} = P_i(t) - D_i(t) - \theta_i(t) I_i(t); \quad 0 \le t \le T_{i1} \qquad (1)$$

$$\frac{dI_i(t)}{dt} = -D_i(t) - \theta_i(t) I_i(t); T_{i1} \le t \le T_{i2} \qquad (2)$$

$$\frac{dI_i(t)}{dt} = -\beta_i(\tau) D_i(t); \; T_{i2} \le t \le T_{i3} \qquad (3)$$

$$\frac{dI_i(t)}{dt} = -(P_i(t) - D_i(t)); \; T_{i3} \le t \le T_{i4} \qquad (4)$$

with the boundary conditions: $I_i(0)$=0, $I_i(T_{i2})$= 0, $I_i(T_{i2})$= 0, $I_i(T_{i4})$ = 0 respectively.

The solutions of the above differential equations under their relative boundary conditions are

$$I_i(t) = e^{-g_i(t)} \int_0^t \{P_i(u) - D_i(u)\} e^{g_i(u)} du, \quad 0 \le t \le T_{i1} \qquad (5)$$

$$I_i(t) = e^{-g_i(t)} \int_t^{T_{i2}} D_i(u) e^{g_i(u)} du; \quad T_{i1} \le t \le T_{i2} \qquad (6)$$

$$I_i(t) = -\int_{T_{i2}}^t \beta_i(\tau) D_i(u) du; T_{i2} \le t \le T_{i3} \qquad (7)$$

$$I_i(t) = -\int_t^{T_{i4}} \{P_i(u) - D_i(u)\} du, T_{i3} \le t \le T_{i4} \qquad (8)$$

respectively, where

$$g_i(t) = \int_0^t \theta_i(u) du \qquad (9)$$

For raw material *j (j=1,2,…,m)*, the changes of the inventory level $q_j(t)$ is governed by the following differential equations

$$\frac{dq_j(t)}{dt} = -r_{ij} P_i(t) - \delta_j(t) q_j(t); \quad 0 \le t \le T_{i1} \qquad (10)$$

$$\frac{dq_j(t)}{dt} = -r_{ij} P_i(t) - \delta_j(t) q_j(t); \; T_{i3} \le t \le T_{i4} \qquad (11)$$

with the boundary conditions $q_j(T_{i1})$=0 , $q_j(T_{i4})$= 0, respectively. The solutions of the last two differential equations under their relative boundary conditions are

$$q_j(t) = r_{ij} e^{-\lambda_j(t)} \int_t^{T_{i1}} e^{\lambda_j(u)} P_i(u) du, \; 0 \le t \le T_{i1} \qquad (12)$$

$$q_j(t) = r_{ij} e^{-\lambda_j(t)} \int_t^{T_{i4}} e^{\lambda_j(u)} P_i(u) du; \; T_{i3} \le t \le T_{i4} \qquad (13)$$

respectively; *j=1,2……,m* where

$$\lambda_j(t) = \int_0^t \delta_j(u) du \;\; ;j=1,2……,m \qquad (14)$$

Next, we derive the present worth of each type of cost and for final product *i* .

Present worth of holding cost for final product i (PWHCFi):

Final product *i* is held in stock in the two periods

[0, T_{i1}] and [T_{i1}, T_{i2}], so we have

$$PWHCFi = \int_0^{T_{i1}} h_i e^{-rt} I_i(t) dt + \int_{T_{i2}}^{T_2} h_i e^{-rt} I_i(t) dt =$$

$$\int_0^{T_{i1}} h_i e^{-rt} [e^{-g_i(t)} (\int_0^t \{P_i(u) - D_i(u)\} e^{g_i(u)} du] dt +$$

$$\int_{T_{i1}}^{T_{i2}} h_i e^{-rt} e^{-g_i(t)} (\int_t^{T_{i2}} D_i(u) e^{g_i(u)} du) dt \; .$$

Integration by parts, we obtain

$$PWHCFi = \int_0^{T_{i1}} [H_i(T_{i1}) - H_i(t)] \{P_i(t) - D_i(t)\} e^{g_i(t)} d\,t$$

$$+ \int_{T_{i1}}^{T_{i2}} [H_i(t) - H_i(T_{i1})] D_i(t) e^{g_i(t)} dt \qquad (15)$$

where

$$H_i(t)=\int_o^t h_i e^{-ru-g_i(u)}du\ , \qquad (16)$$

Present worth of shortage cost for backordered items from final product i (PWSCBFi):

Shortages for any final product i occur in two periods, $[T_{i2},T_{i3}]$ and $[T_{i3},T_{i4}]$ so we have

$$\text{PWSCBFi}=\int_{T_{i2}}^{T_{i3}} b_i e^{-rt} I_i(t)dt+\int_{T_{i3}}^{T_{i4}} b_i e^{-rt} I_i(t)dt$$

$$=\int_{T_{i2}}^{T_{i3}} b_i e^{-rt}(\int_{T_{i2}}^{t}\beta_i(T_{i3}-u)D_i(u)du)dt+$$

$$\int_{T_{i3}}^{T_{i4}} b_i e^{-rt}(\int_{t}^{T_{i4}}\{P_i(u)-D_i(u)\}du)dt$$

Integrating by parts, we get:

PWSCBFi =

$$\int_{T_{i2}}^{T_{i3}}[B_i(T_{i3})-B_i(t)]\beta_i(T_{i3}-t)D_i(t)dt\ +$$

$$\int_{T_{i3}}^{T_{i4}}[B_i(t)-B_i(T_{i3})]\{P_i(t)-D_i(t)dt \qquad (17)$$

where

$$B_i(t)=\int_0^t b_i e^{-ru}du\,, \qquad (18)$$

Present worth of storage cost for lost items from final product i (PWSCLFi):

In a small time period (dt) we lose a fraction $1-\beta_i(T_{i3}-t)]D_i(t)dt$ from the final product i, hence:

$$PWSCLFi=\int_{T_{i2}}^{T_{i3}} l_i e^{-rt}[1-\beta_i(T_{i3}-t)]D_i(t)dt \qquad (19)$$

Using similar techniques we get the following:

Present worth of item production cost for final product i (PWPCFi):

Since production occurs during the two periods $[0, T_{i1}]$ and $[T_{i3},T_{i4}]$, so we have:

$$PWPCFi=\{\int_0^{T_{i1}} c_i P_i(t)e^{-rt}dt+\int_{T_{i3}}^{T_{i4}} c_i P_i(t)e^{-rt}dt\} \qquad (20)$$

Note that the last cost includes both consumed and deteriorated items from the final product i .

Present worth of the set-up cost for final product *i (PWSUCFi):*

.The set-up of new production for final product i occurs twice during any cycle. The first is at

t = 0, and the second is at $t = T_{i4}$. Therefore, the present worth of the set-up cost for final product i is

$$PWSUCFi=s_i e^{-r.0}+s_i e^{-rT_{i3}}=s_i+s_i e^{-rT_{i3}} \qquad (21)$$

Present worth of item profit for final product i (PWNFi)is equal to the present worth of profit of selling the produced items subtracting the present worth of profit of deteriorated items:

As we have indicated above ,the number of produced items is equal to

$$\int_0^{T_{i1}} P_i(t)dt+\int_{T_{i3}}^{T_{i4}} P_i(t)dt\}$$

and the number of deteriorated items is equal to

$$\theta_i(t)\ [\int_0^{T_{i1}} P_i(t)dt+\int_{T_{i3}}^{T_{i4}} P_i(t)dt\}\quad]\ .$$

Hence, we have

$$PWNFi=\int_0^{T_{i1}} \mathrm{e}^{-\mathrm{rt}}\mathrm{n}_{\mathrm{i}}(\mathrm{t})\,P_i(t)[1-\theta_{\mathrm{i}}(\mathrm{t})\,]dt+$$

$$\int_{T_{i3}}^{T_{i4}} \mathrm{e}^{-\mathrm{rt}}\mathrm{n}_{\mathrm{i}}(\mathrm{t})\,P_i(t)[1-\theta_{\mathrm{i}}(\mathrm{t})\,]dtt$$

Hence, the total relevant net profit per unit time for the final product i as a function of $T_{i1},T_{i2},T_{i3},T_{i4}$,which we shall denote by F_i $(T_{i1},T_{i2},T_{i3},T_{i4})$, is given by

$$F_i=\frac{1}{T_{i4}}\{PWNFi-[PWHCFi+PWSCBFi+PWSCBFi+PWSCLFi+PWPCFi+PWSUCFi]\} \quad (22)$$

or

$$F_i=\frac{1}{T_{i4}}\{\int_0^{T_{i1}}\mathrm{e}^{-rt}\mathrm{n}_i(t)\,P_i(t)[1-\theta_i(t)]dt+\int_{T_{i3}}^{T_{i4}}\mathrm{e}^{-rt}\mathrm{n}_i(t)\,P_i(t)[1-\theta_i(t)]dt\}$$
$$-\frac{1}{T_{i4}}\{\int_0^{T_{i1}}[H_i(T_{i1})-H_i(t)][P_i(t)-D_i(t)]e^{g_i(t)}dt+$$
$$\int_{T_{i1}}^{T_{i2}}[H_i(t)-H_i(T_{i1})]D_i(t)e^{g_i(t)}dt+$$
$$\int_{T_{i2}}^{T_{i3}}[B_i(T_{i3})-B_i(t)]\beta_i(T_{i3}-t)D_i(t)dt+$$
$$\int_{T_{i3}}^{T_{i4}}[B_i(t)-B_i(T_{i3})][P_i(t)-D_i(t)]dt$$
$$+\int_{T_{i2}}^{T_{i3}}l_ie^{-rt}[1-\beta_i(T_{i3}-t)]D_i(t)dt+$$
$$[\int_0^{T_{i1}}c_iP_i(t)e^{-rt}dt+\int_{T_{i3}}^{T_{i4}}c_iP_i(t)e^{-rt}dt]+$$
$$s_i+s_ie^{-rT_{i3}}\} \quad (23)$$

Hence, the total relevant net profit for all final products, say F , is given by

$$F=\sum_{i=1}^{n}F_i \quad (24)$$

where F_i is given by (23).

Now, we derive the present worth concerned with all *raw materials* which are used in producing the same final products.

The present worth of the holding cost of raw material j(PWHCRMj),(j=1,2,…,m):

All raw materials are held in stock in the two periods [0, T_{i1}] and [T_{i3},T_{i4}] so we have

$$PWHCRMj=\int_0^{T_{i1}}e^{-rt}o_jq_j(t)dt+\int_{T_{i3}}^{T_{i4}}e^{-rt}o_jq_j(t)dt=$$
$$\int_0^{T_{i1}}\left\{e^{-rt}o_jr_{ij}e^{-\lambda_j(t)}\int_t^{T_{i1}}e^{\lambda_j(u)}P_i(u)du\right\}dt \;.$$
$$+\int_{T_{i3}}^{T_{i4}}\left\{e^{-rt}o_jr_{ij}e^{-\lambda_j(t)}\int_t^{T_{i4}}e^{\lambda_j(u)}P_i(u)du\right\}dt$$

Integrating by parts we obtain

$$PWHCRMj=r_{ij}\int_0^{T_{i1}}[A_j(t)-A_j(0)]P_i(t)e^{\lambda_j(t)}dt$$
$$+r_{ij}\int_{T_{i3}}^{T_{i4}}[A_j(T_{i3})-A_j(t)]P_i(t)e^{\lambda_j(t)}dt \quad (25)$$

where

$$A_j(t)=\int_0^t o_je^{-rt-\lambda_j(t)}dt \quad (26)$$

The present worth of the cost of raw material j (PWCRMj)(j=1,2,….,m)is equal to:

$$PWCRMj=u_jq_j(0)+u_jq_j(T_{i3})=$$
$$u_jr_{ij}e^{-\lambda_j(0)-0.r}\int_0^{T_{i1}}e^{\lambda_j(u)}P_i(t)du+$$
$$u_jr_{ij}e^{-\lambda_j(T_{i3})-rT_{i3}}\int_{T_{i3}}^{T_{i4}}e^{\lambda_j(u)}P_i(t)dt \quad (27)$$

Finally,

The present worth of ordering raw material j (PWOCRMj) (j=1,2,….,m)is given by

$$PWOCRMj=k_je^{-r.0}+k_je^{-rT_{i3}}=$$
$$\boldsymbol{k_j(1+e^{-rT_{i3}})} \quad (28)$$

Thus, the per unit time total relevant cost fro raw material ***j*** as a function of $T_{i1},T_{i2},T_{i3},T_{i4}$ which we shall denote by W_i is given by

$$W_j=\frac{1}{T_{i4}}(PWHCRMj+PWCRMj+PWOCRMj)\;,$$

or

$$W_j = \frac{1}{T_{i4}}\{r_{ij}\int_0^{T_{i1}}[A_j(t)-A_j(0)]P_i(t)e^{\lambda_j(t)}d\,t$$
$$-r_{ij}\int_{T_{i3}}^{T_{i4}}[A_j(T_{i3})-A_j(t)]P_i(t)e^{\lambda_j(t)}d\,t+$$
$$u_j r_{ij}e^{-\lambda_j(0)-0.r}\int_0^{T_{i1}}e^{\lambda_j(u)}P_i(u)du\ +$$
$$u_j r_{ij}e^{-\lambda_j(T_{i3})-rT_{i3}}\int_{T_{i3}}^{T_{i4}}e^{\lambda_j(u)}P_i(u)du\ \}dt+$$
$$k_j(1+e^{-rT_{i3}})\} \qquad (29)$$

Thus, the total relevant cost of all raw materials which we shall denote by W is given by

$$W=\sum_{j=1}^{m}W_j \qquad (30)$$

Now, let $T_1=[T_{i1}]\,,T_2=[T_{i2}]\,,T_3=[T_{i3}]\ \,,T_4=[T_{i4}]\ :i=1,2,....,n$ be the column vectors ,then the total relevant net profit per unit time for the whole integrated production inventory system, as a function of T_1,T_2,T_3,T_4 which we shall denote by *TNU* **(**T_1,T_2,T_3,T_4**)** is given by

$$TNU(T_1,T_2,T_3,T_4)=\sum_{i=1}^{n}TNU_i(T_{i1},T_{i2},T_{i3},T_{i4})$$
$$=\sum_{i=1}^{n}F_i+\sum_{j=1}^{m}W_j \qquad (31-a)$$
$$TNU_i(T_{i1},T_{i2},T_{i3},T_{i4})=F_i+W_j \qquad (31-b)$$

Our problem is to find the optimal values of the vectors T_1,T_2,T_3,T_4 that maximize *TNU* (T_1,T_2,T_3,T_4) given by (31-a) subject to the following constraint:

$$0<T_{i1}<T_{i2}<T_{i3}<T_{i4} \qquad (32)$$
$$F_{i1}:\int_{T_{i2}}^{T_{i3}}\beta_i(T_3-t)D_i(t)dt-\int_{T_{i3}}^{T_{ii4}}[P_i(t)-D_i(t)]dt=0 \qquad (33)$$
$$F_{i2}:\int_0^{T_{i1}}P_i(t)e^{g_i(t)}dt-\int_0^{T_{i2}}D_i(t)e^{g_i(t)}dt=0 \qquad (34)$$

$i=1,2,\dots n$

Note that constraint (32) is a natural constraint since; otherwise, our problem would have no meaning. Constraint (33) comes from the fact that, the inventory levels given by (7) & (8) must be equal at $t=T_{i3}$, whereas constraint (34) comes from the fact that the inventory levels given by (5) & (6) must be equal at $t=T_{i1}$. Note also that, by the two constraints (33) and (34) we attain the continuity of the production process for the final product i. On the other hand, by our above assumption *(AS1.)* we have the following constraints on raw materials:

$$\sum_{j=1}^{m}q_j(0)=\sum_{i=1}^{n}\ \sum_{j=1}^{m}r_{ij}\int_0^{T_{i1}}P_i(t)dt$$

and

$$\sum_{j=1}^{m}q_j(T_{i3})=\sum_{i=1}^{n}\ \sum_{j=1}^{m}r_{ij}\int_{T_{i3}}^{T_{i4}}P_i(t)dt$$

which are, respectively, equivalent to

$$M_{1j}:\sum_{j=1}^{m}\ r_{ij}[\int_0^{T_{i1}}e^{\lambda_j(t)}P_i(t)dt$$
$$-\sum_{i=1}^{n}\ \sum_{j=1}^{m}\ r_{ij}[\int_0^{T_{i1}}P_i(t)dt=0;\ j=1,2,.......,m \qquad (35)$$
$$M_{2j}:\sum_{j=1}^{m}r_{ij}\int_{T_{i3}}^{T_{i4}}e^{\lambda_j(t)-\lambda_j(T_{i3})}P_i(t)dt]$$
$$-\sum_{i=1}^{n}\ \sum_{j=1}^{m}r_{ij}\int_{T_{i3}}^{T_{i4}}P_i(t)dt]=0\ \ j=1,2,.......,m \qquad (36)$$

(35) assure that the availability of all raw materials in the first stage of production shall cover, exactly, the need of the final products from these raw materials , whereas relations (36) guarantee similar condition for the second stage of production. Thus, our problem (call it ***(P)***) is given by

Maximize *TNU* (T_1,T_2,T_3,T_4) subject to (32), (33) , (34) ,(35) &(36) ***(P)***

4. Solution Procedure

To solve problem (P), we first ignore (32). This can be justified by the reasons that; if

(32) does not hold, then the whole problem would have no meaning. However, we shall not consider any solution that does not satisfy (32). Thus, our new problem (call it P_1) is:
Maximize $TNU\,(T_1,T_2,T_3,T_{i4})$ subject to (33), (34), (35) &(36) ***(P_1)***
Note that (P_1) is an optimization problem with *2n+2m* equality constraints, so it can be solved by LagrangeTechniques. Let $L(T_1,T_2,T_3,T_4,\sigma_1,\sigma_2,...,\sigma_n,\gamma_1,\gamma_2,...,\gamma_n,$ $\mu_1,\mu_2,...,\mu_m,\alpha_1,\alpha_2,...,\alpha_m)$ be our Lagrangean, where $\sigma_1,\sigma_2,...,\sigma_n,\gamma_1,\gamma_2,...,\gamma_n,\mu_1,\mu_2,........,\mu_m$ $,\alpha_1,\alpha_2,........,\alpha_m$ are the Lagrange multipliers concerned with the constraints (33) , (34) ,(35) &(36) respectively, then

$$L(T_1,T_2,T_3,T_4,\sigma_1,\sigma_2,...,\sigma_n,\gamma_1,\gamma_2,...,\gamma_n,\mu_1,\mu_2,..,\mu_m,\alpha_1,\alpha_2,..,\alpha_m)$$

$$=TNU(T_1,T_2,T_3,T_4)+\sum_{i=1}^{n}\sigma_i F_{i1}+\sum_{i=1}^{n}\gamma_i F_{i2}+\sum_{j=1}^{m}\mu_j M_{1j}+\sum_{j=1}^{m}\alpha_j M_{2j}$$

Or ,equivalently

$$L(T_1,T_2,T_3,T_4,\sigma_1,\sigma_2,...,\sigma_n,\gamma_1,\gamma_2,...,\gamma_n,\mu_1,\mu_2,..,\mu_m,\alpha_1,\alpha_2,..,\alpha_m)$$

$$=\sum_{i=1}^{n}[TNU_i\,(T_{i1},T_{i2},T_{i3},T_{i4})+\sigma_i F_{i1}+\gamma_i F_{i2}]+\sum_{j=1}^{m}[\mu_j M_{1j}+\alpha_j M_{2j}] \qquad (37)$$

The necessary conditions for having *an optima* are:

$$\frac{\partial L}{\partial T_{i1}}=0,\frac{\partial L}{\partial T_{i2}}=0,\frac{\partial L}{\partial T_{i3}}=0,\frac{\partial L}{\partial T_{i4}}=0,$$
$$i=1,2,...n \qquad (38)$$

$$\frac{\partial L}{\partial \sigma_i}=0,\ \frac{\partial L}{\partial \gamma_i}=0,\ \frac{\partial L}{\partial \mu_j}=0,\ \frac{\partial L}{\partial \alpha_j}=0$$

$$i=1,2,\ldots,n,\ and\ \ j=1,2,...,m \qquad (39)$$

Note that relations (39) repeat the constraints (33) through (36) respectively. Here we have

$$\frac{\partial L}{\partial t}=\sum_{i=1}^{n}\frac{\partial F_i}{\partial t}+\sum_{j=1}^{m}\frac{\partial W_j}{\partial t}+\sum_{i=1}^{n}\sigma_i\frac{\partial F_{i1}}{\partial t}+\sum_{i=1}^{n}\gamma_i\frac{\partial F_{i2}}{\partial t}+\sum_{j=1}^{m}\mu_j\frac{\partial M_{1j}}{\partial t}+\sum_{j=1}^{m}\alpha_j\frac{\partial M_{2j}}{\partial t} \qquad (40)$$

Recalling that, for a specific product k , we are seeking the optimal values of $T_{k1},T_{k2},T_{k3},T_{k4}$,so we have nothing to do with the partial derivatives in (40) other than those concerned with the product *k*. Thus we get the following results .

For *i=1,2,…,n* ,we have:

From(40)

$$\frac{\partial L}{\partial T_{i1}}=0\Rightarrow(\sin ce\ P_i(T_{i1})>0, And\,by(34))$$

$$\frac{1}{T_{i4}}\{e^{-rT_{i1}}\mathrm{n_i}(T_{i1})[1-\theta_{\mathrm{i}}(T_{i1})]-e^{-rT_{i1}}\mathrm{n_i}(T_{i1})[1-\theta_{\mathrm{i}}(T_{i1})]\}-\frac{1}{T_{i4}}\{c_i e^{-rC}+\sum_{j=1}^{m}r_{ij}e^{\lambda_j(T_{i1})}[(A_j(T_{i1})-A_j(0))+u_j]\}+\sum_{i=1}^{n}\gamma_i e^{g_i(T_{i1})}+\sum_{j=1}^{m}\mu_j(r_{ij}e^{\lambda_j(T_{i1})}-\sum_{i=1}^{n}r_{ij})=0 \qquad (41)$$

$$\frac{\partial L}{dT_{i2}}=0\Rightarrow(\sin ce\,D_i(T_{i2})>0)$$

$$\frac{1}{T_{i4}}\{e^{-rT_{i2}}\mathrm{n_i}(T_{i2})[1-\theta_{\mathrm{i}}(T_{i2})]\}-\frac{1}{T_{i4}}\{[H_i(T_{i2})-H_i(T_{i1})]e^{g_i(T_{i2})}=[B_i(T_{i3})-B_i(T_{i2})]\beta_i(T_{i3}-T_{i2})+e^{-rT_{i2}}l_i[1-\beta_i(T_{i3}-T_{i2})]\}+\sum_{i=1}^{n}[\sigma_i\beta_i(T_{i3}-T_{i2})+\gamma_i e^{g_i(T_{i2})}] \qquad (42)$$

$$\frac{\partial L}{\partial T_{i3}}=0\Rightarrow -\frac{1}{T_{i4}}\{\int_{T_{i2}}^{T_{i3}}\beta_i(T_{i3}-t)[B_i(T_{i3})$$
$$-B_i(t)]D_i(t)dt-[\int_{T_{i2}}^{T_{i3}}e^{-rt}l_i\beta_i(T_{i3}-t)D_i(t)dt$$
$$+c_ie^{-rT_{i3}}P_i(T_{i3})+rs_ie^{-rT_{i3}}]-$$
$$\sum_{j=1}^{m}r_{ij}u_j(\lambda_j'(T_{i3})+r)e^{-\lambda_j(T_{i3})-rT_{i3}}\int_{T_{i3}}^{T_{i4}}P_i(t)e^{\lambda_j(t)}dt$$
$$+\sum_{j=1}^{m}r_{ij}u_je^{-rT_{i3}}P_i(T_{i3})+re^{-rT_{i3}}\sum_{j=1}^{m}k_j\}$$
$$-\sum_{i=1}^{n}\sigma_i[\int_{T_{i2}}^{T_{i3}}\beta_i(T_{i3}-t)D_i(t)dt-P_i(T_{i3})]$$
$$-P_i(T_{i3})\sum_{j=1}^{m}\alpha_j[r_{ij}-\sum_{i=1}^{n}r_{ij}]$$
$$-\sum_{j=1}^{m}\alpha_jr_{ij}\lambda_j'(T_{i3})\int_{T_{i3}}^{T_{i4}}P_i(t)e^{\lambda_j(t)-\lambda_j(T_{i3})}dt=0 \quad (43)$$

$$\frac{\partial L}{\partial T_{i4}}=0\Rightarrow$$
$$\frac{1}{T_{i4}}\{e^{-rT_{i4}}\mathrm{n}_i(T_{i4})[1-\theta_i(T_{ii4})]\}$$
$$-\frac{1}{T_{i4}}\{[B_i(T_{i4})-B_i(T_{i3})][P_i(T_{i4})-D_i(T_{i4})]+$$
$$c_ie^{-rT_{i4}}P_i(T_{i4})+\sum_{j=1}^{m}r_{ij}e^{\lambda_j(T_{i4})}(A_j(T_{i4})-A_j(T_{i3}))]$$
$$+P_i(T_{i4})\sum_{j=1}^{m}r_{ij}u_je^{\lambda_j(T_{i4})-\lambda_j(T_{i3})-rT_{i3}}\}$$
$$-\frac{1}{T_{i4}}TNU_i-\sum_{i=1}^{n}\sigma_i[P_i(T_{i4})-D_i(T_{i4})]$$
$$+P_i(T_{i4})\sum_{j=1}^{m}\alpha_j(r_{ij}e^{\lambda_j(T_{i4})-\lambda_j(T_{i3})}-\sum_{i=1}^{n}r_{ij})=0 \quad (44)$$

which gives

$$TNU_i=\{e^{-rT_{i4}}\mathrm{n}_i(T_{i4})[1-\theta_i(T_{i4})]\}-$$
$$\{[B_i(T_{i4})-B_i(T_{i3})][P_i(T_{i4})-D_i(T_{i4})]+$$
$$c_ie^{-rT_{i4}}P_i(T_{i4})+\sum_{j=1}^{m}r_{ij}e^{\lambda_j(T_{i4})}(A_j(T_{i4})-A_j(T_{i3}))]$$
$$+P_i(T_{i4})\sum_{j=1}^{m}r_{ij}u_je^{\lambda_j(T_{i4})-\lambda_j(T_{i3})-rT_{i3}}\}$$
$$-T_{i4}\sum_{i=1}^{n}\sigma_i[P_i(T_{i4})-D_i(T_{i4})]$$
$$+T_{i4}P_i(T_{i4})\sum_{j=1}^{m}\alpha_j(r_{ij}e^{\lambda_j(T_{i4})-\lambda_j(T_{i3})}-\sum_{i=1}^{n}r_{ij})=0 \quad (45)$$

Note that, (45) gives the maximum total net profit TNU_i of the final product *i* in terms of $\sigma_i,\alpha_i,T_{i3}\,\&\,T_{i4}$.

Equations(33),(34),(35),(36),(41),(42),(43),(44) &(45) are *6n +2m* equations with *6n +2m* variables, namely T_1,T_2,T_3,T_4, $\sigma_1,\sigma_2,...,\sigma_n,\gamma_1,\gamma_2,...,\gamma_n,\mu_1,\mu_2,...,\mu_m,\alpha_1,\alpha_2,...,\alpha_m$ so that the solution(s) of these equations (if exists) gives the critical points of $L(T_1,T_2,T_3,T_4,$ $\sigma_1,\sigma_2,...,\sigma_n,\gamma_1,\gamma_2,...,\gamma_n,\mu_1,\mu_2,...,\mu_m,\alpha_1,\alpha_2,...,\alpha_m)$ from which (T_1,T_2,T_3,T_4) is the corresponding critical point of $TNU(T_1,T_2,T_3,T_4)$.

5. Maximality Uniqueness and Global Optimality of Solutions

In this section, we derive conditions that guarantee the *maximalism* of the solution to problem *(P) such solution exists* by deriving conditions for which the *Hessian Matrix* of our Lagrangian L(.) is negative semi-definite . To compute the Hessian matrix of L(.) we consider the following notations

$$\frac{\partial^2 L}{\partial T_{il}^{\ 2}}=L_{T_{il}^2}\ ,\quad \frac{\partial^2 L}{\partial T_{il}\partial T_{jl}}=L_{T_{il}T_{jl}}\ ,$$

i,j=1,2,……,n and l=1,2,3,4

Then the related computations showed that *for i=1,2,…,n* , L($T_{i1},T_{i2},T_{i3},T_{i4}$) has he following form

$$L(T_{i1}^{*}, T_{i2}^{*}, T_{i3}^{*}, T_{i4}^{*}) =$$

$$\begin{bmatrix} L_{T_{i1}^2} & L_{T_{i1}T_{i2}} & 0 & L_{T_{i1}T_{i4}} \\ L_{T_{i1}T_{i2}} & L_{T_{i2}^2} & L_{T_{i2}T_{i3}} & 0 \\ 0 & L_{T_{i2}T_{i3}} & L_{T_{i3}^2} & L_{T_{i3}T_{i4}} \\ L_{T_{i1}T_{i4}} & 0 & L_{T_{i3}T_{i4}} & L_{T_{i4}^2} \end{bmatrix} \quad (46)$$

Where $T_{i1}^{*}, T_{i2}^{*}, T_{i3}^{*}, T_{i4}^{*}$ are the optimal values of $T_{i1},T_{i2},T_{i3},T_{i4}$.

By Balkhi and Bebkherouf [13], Stewart [35], this symmetric matrix is negative semi-definite if

$$L_{T_{i1}^2} \geq \left|L_{T_{i1}T_{i2}}\right| + \left|L_{T_{i1}T_{i4}}\right| \quad (47)$$

$$L_{T_{i2}^2} \geq \left|L_{T_{i1}T_{i2}}\right| + \left|L_{T_{i2}T_{i3}}\right| \quad (48)$$

$$L_{T_{i3}^2} \geq \left|L_{T_{i2}T_{i3}}\right| + \left|L_{T_{i3}T_{i4}}\right| \quad (49)$$

$$L_{T_{i4}^2} \geq \left|L_{T_{i1}T_{i4}}\right| + \left|L_{T_{i3}T_{i4}}\right| \quad (50)$$

Note that if (47) through (50) hold for each *i; i=1,2,…,n* , then the corresponding total relevant net profit for the final product *i* would have a maximum value which in turn implies that the total relevant net profit for the whole integrated production inventory system ,considered here , is also maximum .Thus, the above arguments lead to the following theorem.

Theorem 1. *Any existing solution of (P_1) is a maximizing solution to (P_1) if this solution satisfies (47) through (50).*

On the other hand, any existing and maximizing solution of *(P_1)* is unique, hence it is global optimal. To see this, we note, from (33),(34),(35),(36),(41),(42),(43),(44),&(45) that each of $T_{i1},T_{i2},T_{i3},T_{i4}$ can implicitly be determined as a function of, *say* T_{i1} i.e $T_{i1} = f_1(T_{i1}), T_{i2} = f_2(T_{i1}), T_{i3} = f_3(T_{i1}), T_{i4} = f_4(T_{i1})$ from which and relations (41),(42),(43),(44) &(45) we can also conclude that each of $\sigma_i,\gamma_i,\mu_j,\alpha_j$ is also a function of *T_{i1}*, say $\sigma_i = \sigma_i(T_{i1})$, $\gamma_i = \gamma_i(T_{i1}), \mu_j = \mu_j(T_{i1}), \alpha_j = \alpha_j(T_{i1})$ .Our arguments in showing the uniqueness and global optimality of the solution is based on the idea that *the general value of TNU_i given by (31-b) must **coincide** with the maxium value of TNU_i given by(45).* That is

$$V_i\,(T_{i1}, f_2(T_{i1}), f_3(T_{i1}), f_4(T_{i1}))/ f_4(T_{i1}) -$$
$$TNU_i\,(\sigma_i(T_{i1}), \alpha_j(T_{i1}), f_3(T_{i1}), f_4(T_{i1}),)) = 0 \quad (51)$$

Where $\underline{TNU_i = V_i} / f_4(T_{i1})$

Here, $V_i(T_{i1}, f_2(T_{i1}), f_3(T_{i1}), f_4(T_{i1}))/ f_4(T_{i1})$ is taken from (31-b) and TNU_i $(\sigma_i(T_{i1}),\alpha_j(T_{i1}), f_3(T_{i1}), f_4(T_{i1})))$ is taken from (45).

Note that any maximizing solution of *(P_1)* (if it exists) is unique (hence global maximum) if equation (51), as an equation of T_{i1} , has a unique solution. This fact has been shown by Balkhi([5] ,[7] &[12]) .

Thus , the above arguments lead to the following theorem

Theorem 2. *Any existing solution of (P_1) for which (47) through (50) hold, is the unique and global optimal solution to (P_1) .*

6. Conclusion

In this paper, we have considered a unified and general multi item integrated production inventory system for which the economic order quantity (EOQ) of raw materials, which are used in producing several final products, as well as the economic production quantity (EPQ) of these products are treated together .Each of the demand, production and deterioration of the final products vary arbitrarily with time, whereas the deterioration rates of the required raw materials are also arbitrarily time varying .Each of the required raw material ,needed to produce one unit of any of the final products, is proportional to the production rate of this final product. Shortages are allowed only for the final products but are partially backordered. Both inflation and time value of money are incorporated in all costs and profits components. The objective is to maximize the overall total relevant net profit of this unified and multi item integrated production inventory system. We have built an exact mathematical model for such unified multi item system ,and introduced a solution procedure by which we can determine the optimal stopping and restarting production times in any cycle for any of the final products. Then, sufficient conditions that guarantee the uniqueness and global optimality of any existing solution are established. Most of related models that have been introduced by previous authors are special cases of our model. This seems to be the first time where such unified and multi item integrated production inventory system is mathematically treated.

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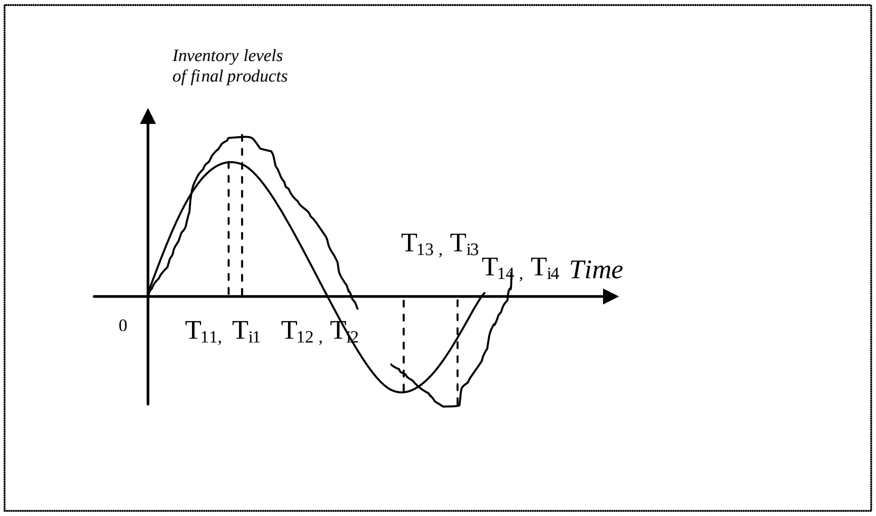
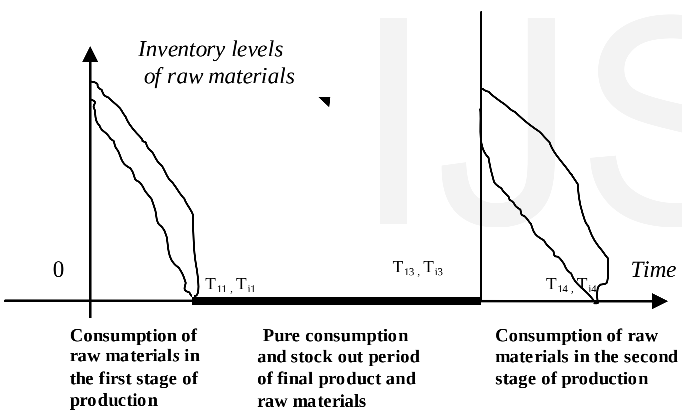
Fig. 1


Inventory levels
of final products
T13 , Ti3
T14 , Ti4 Time
0
T11, Ti1 T12 , Ti2


Inventory levels
of raw materials
0
T11 , Ti1
T13 , Ti3
T14 , Ti4
Time
Consumption of raw materials in the first stage of production
Pure consumption and stock out period of final product and raw materials
Consumption of raw materials in the second stage of production